On The Superposition Theorem For Linear Networks

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Below is a voltage source version of The Superposition Theorem For Linear Networks which states that the current through any given element of a linear network is equal to the sum of the currents due to each voltage source with all other voltage sources set to 0. Here the linear network is assumed to consist entirely of resistors and DC voltage sources. The Solution Principle For Linear Networks and Superposition Theorem For Linear Networks also holds when current sources are included, and more generally when AC voltage sources, inductors, and capacitors, are included.

The Solution Principle For Linear Networks If a linear network consists entirely of resistors and constant voltage sources, then using KVL and KCL a system of n linear equations in n unknowns can be constructed such that the unknowns of the system correspond to all the currents of the network, the constant terms of the system are sums and differences of the voltage sources, each voltage source occurs in at least one constant term, the system has a unique solution, and if any other such system of n linear equations in n unknowns has another unique solution, then the other unique solution is identical to the original unique solution.

It is not known if The Solution Principle For Linear Networks is true or false. It has been tested thousands, if not hundreds of thousands, of times and always found to be true.

The Superposition Theorem For Linear Networks. If a linear network consists entirely of resistors and constant voltage sources, then there exists a system of n equations in n unknowns

$$a_{11} \, i_1 + ... + a_{1n} \, i_n = b_{11} \, V_1 + ... + b_{1m} \, V_m$$
$$...$$
$$a_{n1} \, i_1 + ... + a_{nn} \, i_n = b_{n1} \, V_1 + ... + b_{nm} \, V_m$$

where the a's, b's, and V's are numbers, the V's correspond to all of the voltage sources, the i's are variables which correspond to all currents of the network, and there is a unique solution $(s_1, ... , s_n)$ of the system. And if for each $k = 1 , ... , m$ a new linear network is derived from the original network by setting all voltage sources except V_k equal to 0, then for each of these new linear networks there is a system of n equations in n unknowns

$$c^k_{11} \, i_1 + ... + c^k_{1n} \, i_n = d_{1k} \, V_k$$
$$...$$
$$c^k_{n1} \, i_1 + ... + c^k_{nn} \, i_n = d_{nk} \, V_k$$

which has a unique solution $(s_{k1}, ... , s_{kn})$, and for $j = 1 , ... , n$, $s_j = s_{1j} + ... + s_{mj}$.

Proof. By the Solution Principle For Linear Networks there exists a system of n equations in n unknowns such that the unknowns of the system i_j correspond to all of the currents of the network, the constant terms of the system are linear combinations of the voltage sources, and the system has a unique solution. Let us denote this system by

$$a_{11} \, i_1 + ... + a_{1n} \, i_n = b_{11} \, V_1 + ... + b_{1m} \, V_m$$
$$...$$
$$a_{n1} \, i_1 + ... + a_{nn} \, i_n = b_{n1} \, V_1 + ... + b_{nm} \, V_m \, .$$

and its unique solution by $(s_1, ... , s_n)$. Since the solution of the system is unique, the determinant of the

coefficient matrix is not 0, i.e., $|A| \neq 0$.

By Cramer's Rule, for $j = 1, ..., n$,

$$s_j = \left((b_{11} V_1 + ... + b_{1m} V_m) |A_{1j}| + ... + (b_{n1} V_1 + ... + b_{nm} V_m) |A_{nj}| \right) / |A| ,$$

where A_{ij} is the ij-th cofactor of A, thus

$$s_j = \left(b_{11}V_1 |A_{1j}| + ... + b_{n1}V_1 |A_{nj}| \right) / |A| + ... + \left(b_{1m}V_m |A_{1j}| + ... + b_{nm}V_m |A_{nj}| \right) / |A| .$$

If we let $t_{ij} = \left(b_{1i}V_i |A_{1j}| + ... + b_{ni}V_i |A_{nj}| \right) / |A|$, then

$$s_j = t_{1j} + ... + t_{mj} .$$

Also, it follows that for $k = 1, ... , m$,

$(t_{k1}, ... , t_{kn})$ is a solution of

$a_{11} i_1 + ... + a_{1n} i_n = b_{1k} V_k$

...

$a_{n1} i_1 + ... + a_{nn} i_n = b_{nk} V_k$.

Since $|A| \neq 0$, the solution is unique in each case. Note that for $k = 1 , ... , m$ the k-th linear system of n equations in n unknowns above is associated with the linear network which is the same as the original network except that all voltage sources are 0 except V_k. By The Solution Principle For Linear Networks, for each k there exists a system of n linear equations in n unknowns

$c^k_{11} i_1 + ... + c^k_{1n} i_n = d_{1k} V_k$

...

$c^k_{n1} i_1 + ... + c^k_{nn} i_n = d_{nk} V_k$

corresponding to the k-th network, and the system has a unique solution which we will denote by

$(s_{k1}, ... , s_{kn})$.

Also by The Superposition Principle, for $k = 1 , ... , m$,

$(s_{k1}, ... , s_{kn}) = (t_{k1}, ... , t_{kn})$.

So it follows that for $j = 1 , ... , n$,

$$s_j = s_{1j} + ... + s_{mj}.$$

The Solution Principle For Linear Networks does not seem to have been stated before, but it has been used tacitly many times before. The Superposition Theorem is often stated vaguely so that it is unclear what is being superimposed. While I was discussing what is being superimposed with Dr. George Butler, he made this vague situation clear to me. What is being superimposed are linear networks obtained from a given linear network by setting all but one voltage source equal to 0, and each sum of superimposed currents are equal to the corresponding current of the given network.